Network Science Center at West Point

# A Mathematical Model of Network Communication

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# Outline of Presentation

- Motivation
- Communication Network
- Discrete Network Model
- Discrete Conservation of Packets Equations
- Continuum Network Model
- Example: One Dimensional Flow Model
- Current Work

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# Motivation

- Rapid Communication is essential in today's world.
- Understand the dynamics of flow by creating a communication network model.
- Modeling flow on a communication network will allow us to:
  - Describe normal and congested flow on large communication networks.
  - Predict changes in flow pattern due to changes in the spatial density and per-link traffic.

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# Communication Networks

- A communication network is a global system of interconnected networks, both big and small.
- Packet switching network because all data traffic is broken down into data chunks called packets.
- Everything traveling on a communication network is called a packet.

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# Discrete Communication Network

- Use graph theory to describe the connectivity of a network, where a graph is composed of nodes and links.
- Information travels along links connecting nodes.
- The graph is undirected; information travels in both directions.
- The nodes are routers in this model
- Routers will act as both host and switch computers.
- Host computers is where information enters (source), and exits (destination) a network.

# Route Matrix

- The Route matrix depicts the global state of a network.
- Describes how to direct packets from source to destination.
- Each entry describes the next appropriate router a packet will take along its path.
- Routes are pre-determined by an optimal path algorithm.

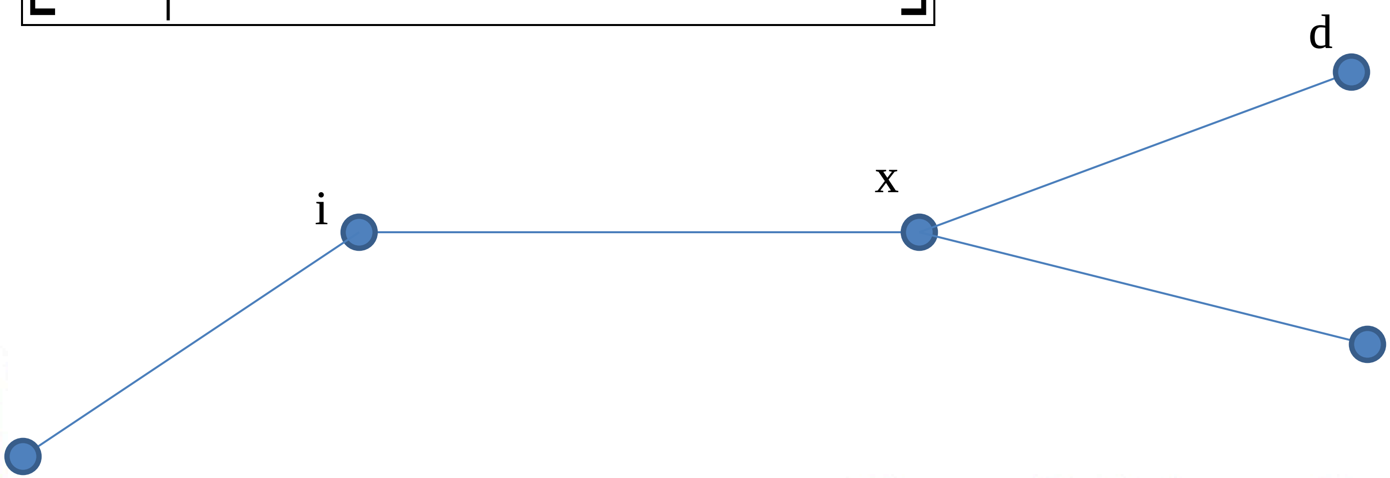
$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & \cdots \end{matrix} \\
\hline
\begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \end{matrix} &
\begin{bmatrix}
r_{1,1} & r_{1,2} & r_{1,3} & \cdots \\
r_{2,1} & r_{2,2} & r_{2,3} & \cdots \\
r_{3,1} & r_{3,2} & r_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{array}
$$

$$r_{i,d} = x$$

$i =$ Current position

$d =$ Destination

$x =$ Next position

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# Queue Dynamics

- Each router contains a queue of buffered packets.
- FIFO unlimited memory buffer
  - Packets enter queue in one of two ways
    - Packets flowed from another router along a connecting link.
    - Generated at a router according to some packet generator rate appropriate to the router.
- Upon being generated, a packet is given a destination by its originator.
- Once a packet reaches its destination it is delivered and exits immediately.

# Flow equation (Arrivals Age zero)

$$b(j,d,0,\tau+1)=\sum_{a=0}^{\infty}\sum_{\substack{i=1\\ j\neq d}}^{N}\delta_{j,R_{i,d}}\beta(i,d,a,\tau)+\nu(j,d,\tau)$$

$b(j,d,a,\tau)$, number of buffered packets at node $j$, with destination $d$, age $a$ at time $t$

$\beta(i,d,a,\tau)$ is the sending rate of packets being sent from router $i$ to router $j$ toward its destination $d$ at time $\tau$

$\nu(j,d,\tau)$ is the number of new packets entering the network at node $j$

# Evolution Equation

- Describes how packets are aging in the buffer.

$$b(j,d,a+1,\tau+1) = b(j,d,a,\tau) - \beta(j,d,a,\tau)$$

$\beta(j,d,a,\tau)$ determines the number of packets that are sent outward from node $j$ with destination $d$

# Discrete Conservation of Packet Equation

- The number of buffered messages at router *j* with destination *d* at time *t*

$$n(j,d,\tau) = \sum_{a=0}^{\infty} b(j,d,a,\tau)$$

$$n(j,d,\tau+1) - n(j,d,\tau) = \sum_{a=0}^{N} b(j,d,a,\tau+1) - \sum_{a=0}^{N} b(j,d,a,\tau)$$

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# Discrete Conservation of Packet Equations

$$n(j,d,\tau+1)-n(j,d,t)=-\sum_{a=0}^{\infty}\beta(j,d,a,\tau)+\sum_{a=0}^{\infty}\sum_{i=1}^{N}\delta_{i,R_{i,d}}\beta(i,d,a,\tau)$$

$$+\nu(j,d,\tau)-\sum_{a=0}^{\infty}\beta(j,j,a,\tau)$$

# Continuum Network Model

- To view this model as a flow model, well discuss the collection of routers as opposed to one. The collection of routers create a Voronoi Diagram.

# Voronoi Diagram

- The spatial location of each router is of importance.
- Each router serves a particular coverage area of users who are sending and receiving packets in an area
- Associate each router with a physical spatial location
- Each router shares physical boundaries with another set of routers so that packets moving through the network will pass through physical boundaries around routers.

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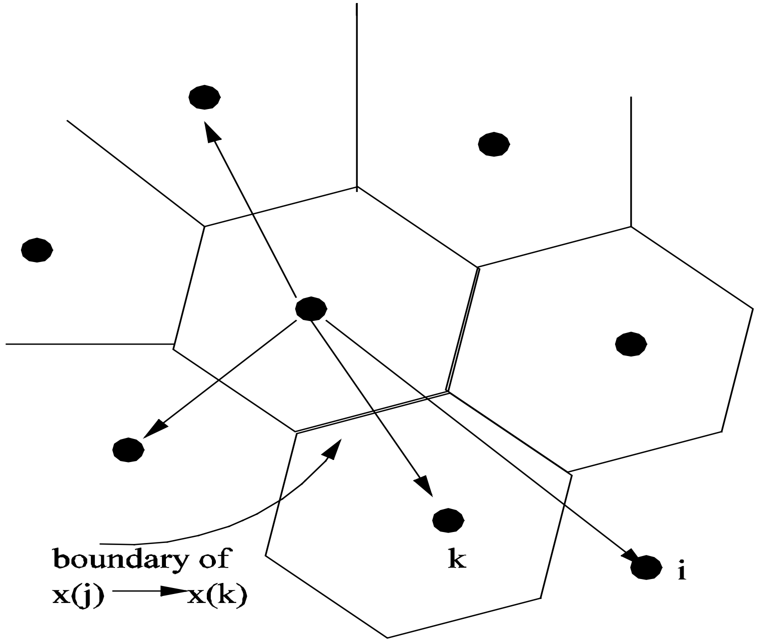
# Voronoi Diagram

$x_j$ = Location of router $j$

$y$ = Location of destination d




Each $x_j$ generates a Voronoi polygon $V(x_j)$, a tessellation in the plane. The Voronoi polygon $V(x_j)$ is the set of points closer to the point (router) $x_j$, than to all other routers in the diagram. A collection of Voronoi polygons is called a Voronoi Diagram.

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# Continuum Description

Let $V_P$ be a Voronoi Diagram and $\partial V_P$ be its boundary

The density of packets for Voronoi polygon $V(x_j)$

$$\rho(x_j, y, t) = \frac{n(j, d, \tau)}{\left|V(x_j)\right|}$$

The number of packets buffered in the Vorono Diagram $V_P$

$$\sum_{j \in P} n(j, d, \tau)$$

# Continuum Description Cont.

The evolution equation for the density of packets in a Voronoi Diagram

$$\int_{V_P} \frac{\rho(x,y,t+\Delta t)-\rho(x,y,t)}{\Delta t}\,dV = \sum_{j\in P}\frac{n(j,d,\tau+1)-n(j,d,\tau)}{\Delta t}$$

Take the limit as $\Delta t$ goes to zero, first term becomes

$$\int_{V_P} \frac{\partial \rho}{\partial t}\,dV$$

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# Flux

The flow vector from router $x_j$ to router $x_i$ is

$$\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(j,d,a,\tau)$$

The outflow through boundary element $\partial V_{l,m}$

$$\sum_{a=0}^{\infty}\sum_{\substack{j\in P\\ i\notin P}}\delta_{(j,i)|(l,m)}\delta_{i,R_{j,d}}\frac{-1}{\Delta t}\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(j,d,a,\tau)\cdot n_{l,m} = \Phi^{O}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

Similar derivation fo the incoming flow terms

$$\sum_{a=0}^{\infty}\sum_{\substack{j\in P\\ i\notin P}}\delta_{(j,i)|(l,m)}\delta_{j,R_{i,d}}\frac{1}{\Delta t}\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(i,d,a,\tau)\cdot n_{l,m} = \Phi^{I}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

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# Flux Cont.

The total flux of packets entering and exiting a boundary $\partial V_{l,m}$

$$\Phi(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right| = \Phi^{O}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right| - \Phi^{I}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

Total flux of packets in Voronoi Diagram $V_P$ with destination $y$

$$\sum_{\substack{l\in\partial P\\ m\notin\partial P}} \Phi(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

Written in the continuum limit and using the divergence theorem the total flux becomes

$$\sum_{\substack{l\in\partial P\\ m\notin\partial P}} \Phi(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right| \rightarrow \int_{\partial V_P} \Phi(x,y,t)\cdot n\, ds$$

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# Source and Sink

The source and sink in their continuum limit

$$\sum_{j \in P} \frac{1}{\Delta t} \nu(j, d, \tau) \rightarrow \int_{V_P} \gamma(x, y, t)\, dv$$

$$\sum_{a=0}^{\infty} \sum_{j \in P} \frac{1}{\Delta t} \beta(j, d, a, \tau) \rightarrow \int_{V_P} \sigma(x, t)\, dV$$

# Continuity Equation

Putting the continuum limits together we have the conservation of packets equation in a Voronoi Diagram $V_P$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \Phi(x, y, t) = \gamma(x, y, t) - \sigma(x, t)$$

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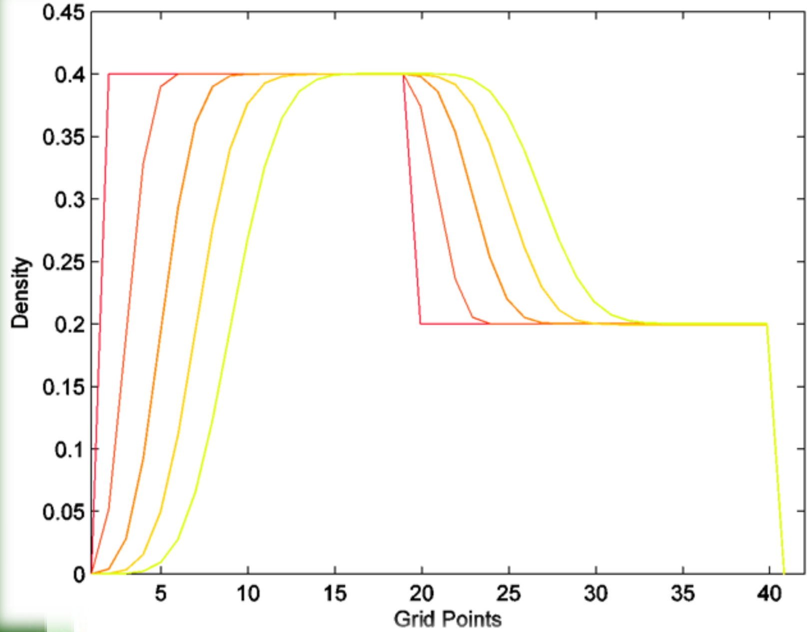
# One Dimensional Network Flow Model

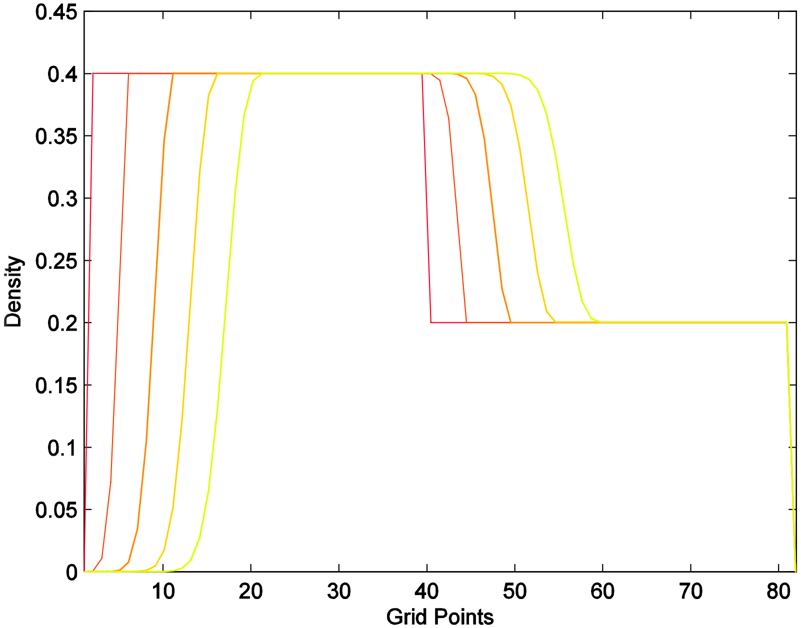
- One dimensional Network flow model with x=0 and a destination x=y.
- Analyze inner nodes
- Continuity equation in one dimensions

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0$$

$$\Phi(x,y,t) = \begin{cases} \rho(x,y,t) & \text{if } \rho(x,y,t) < \Phi_{\max}(x,y,t) \\ \Phi_{\max}(x,y,t) & \text{if } \rho(x,y,t) \geq \Phi_{\max}(x,y,t) \end{cases}$$

# Non-Saturated Flow




Flow Movement In Time

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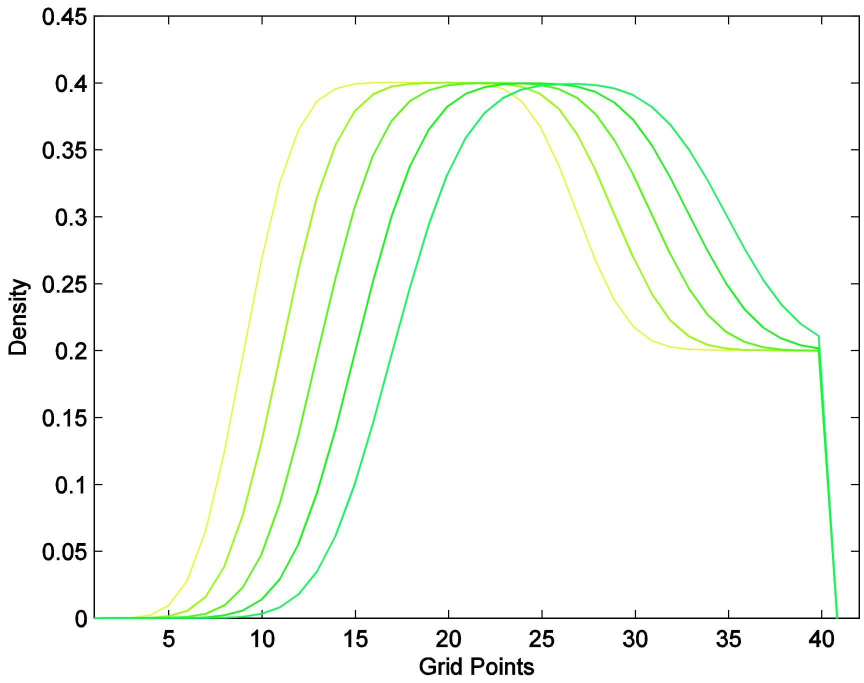


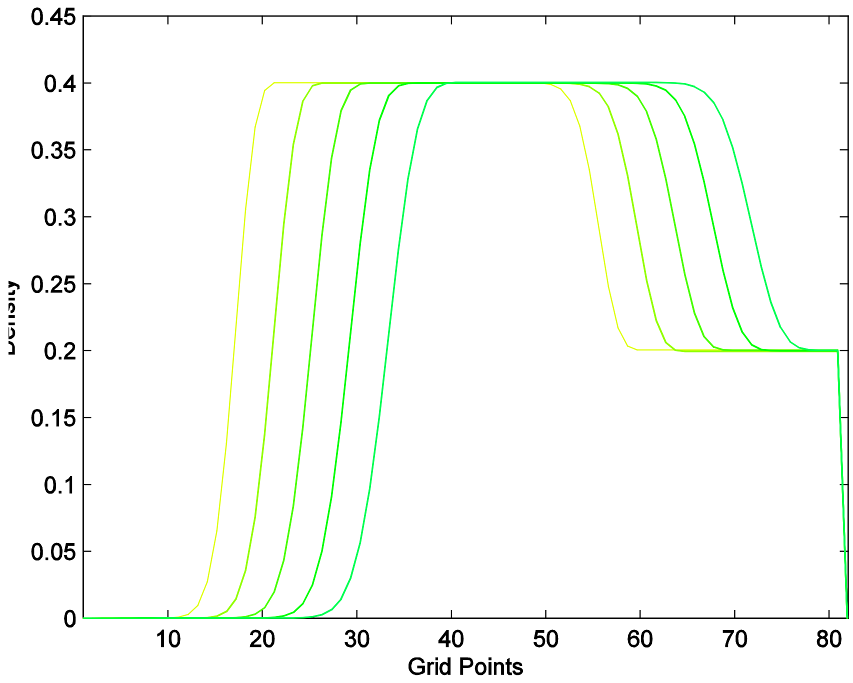


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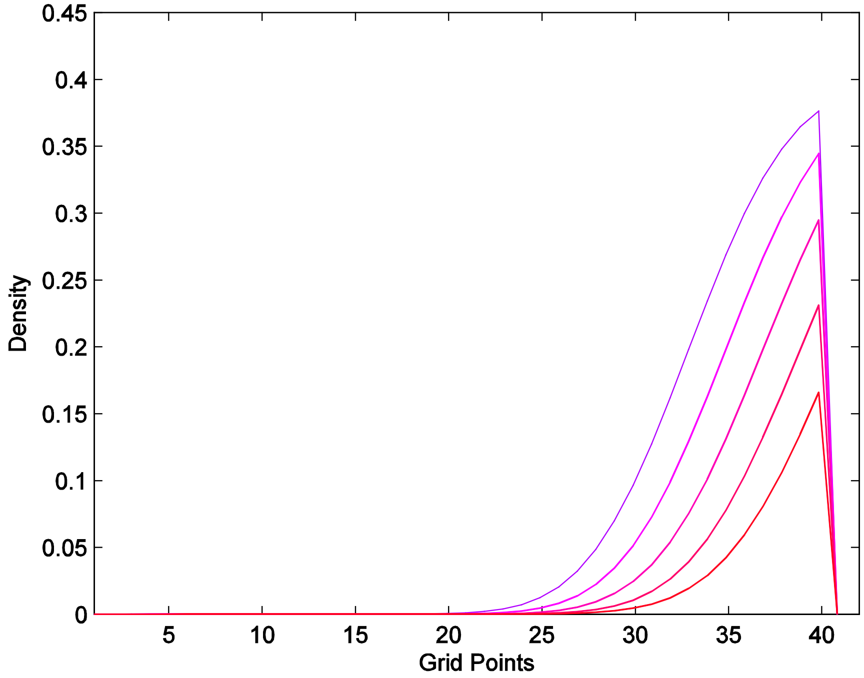
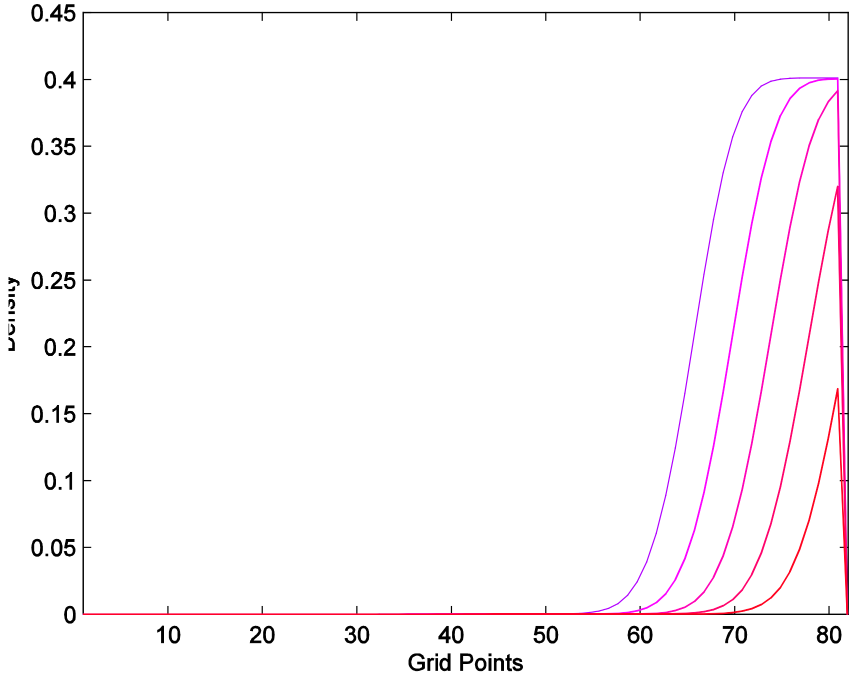
# End of Flow

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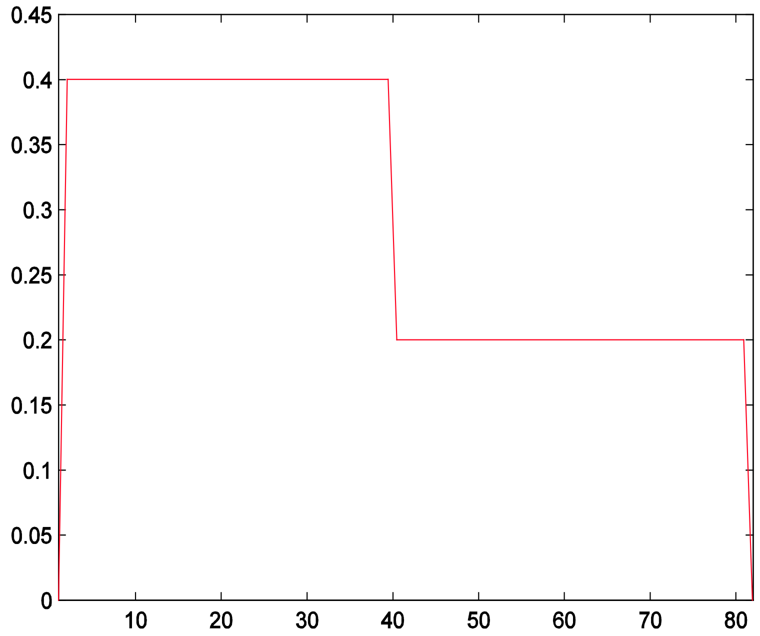
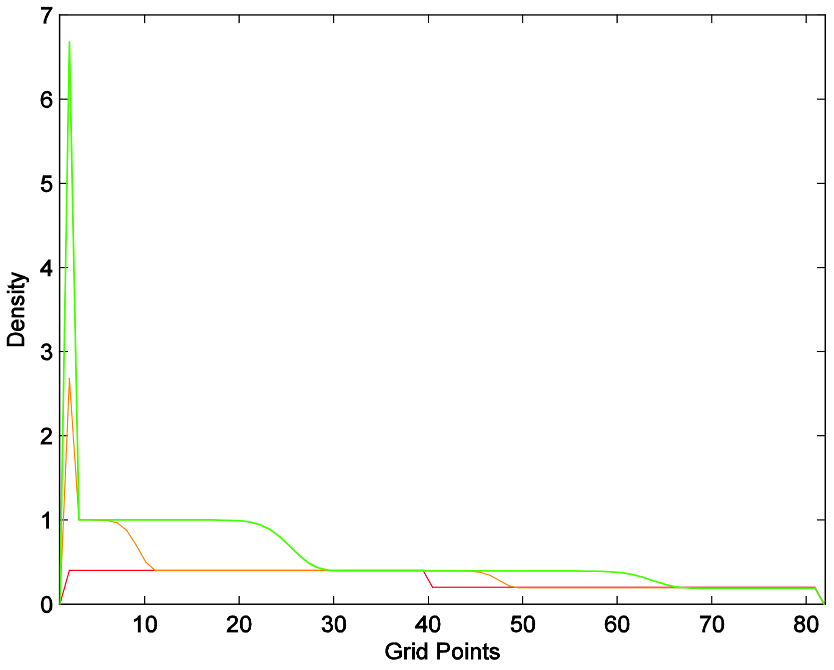
# Example: Flow with Saturation




Initial Condition

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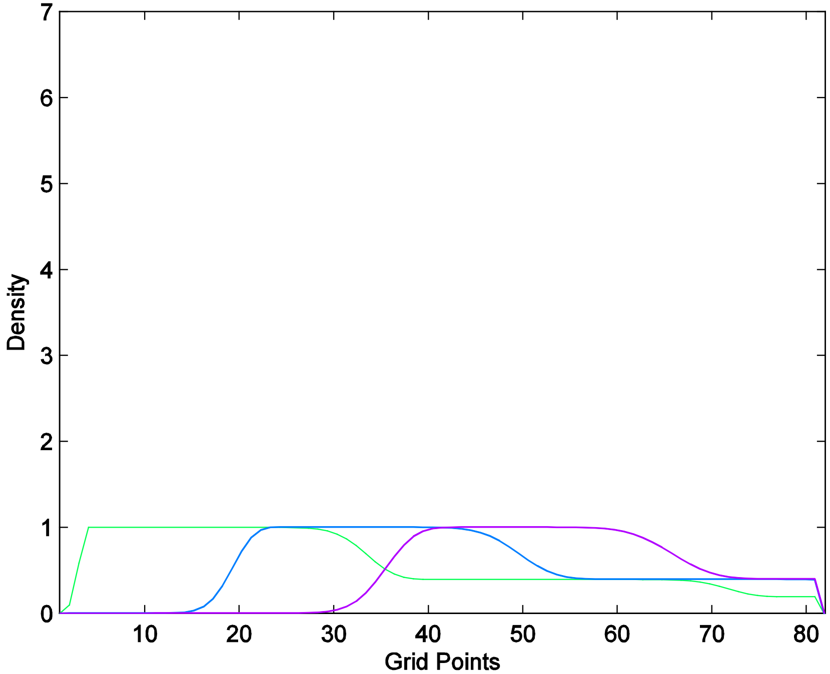
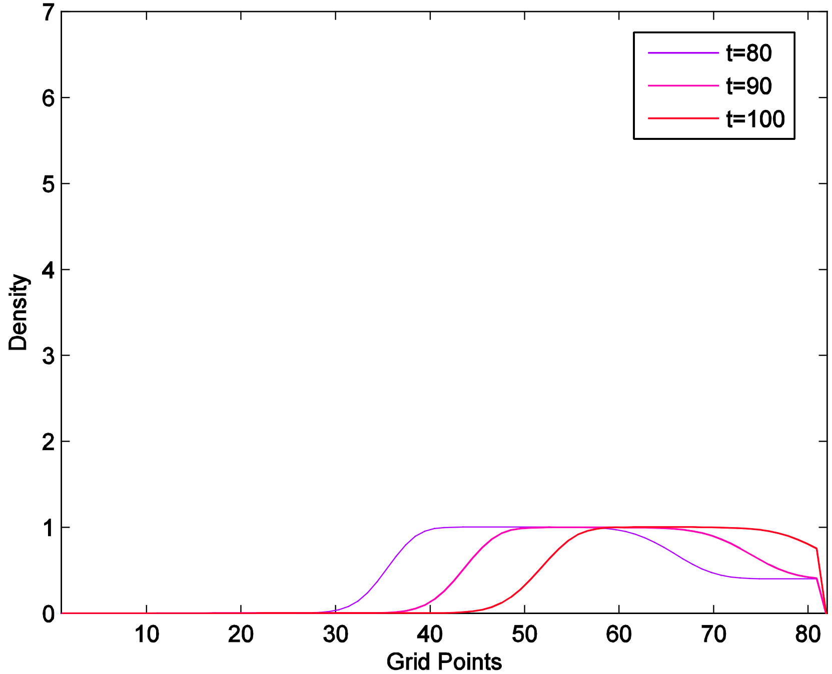
# End of Flow

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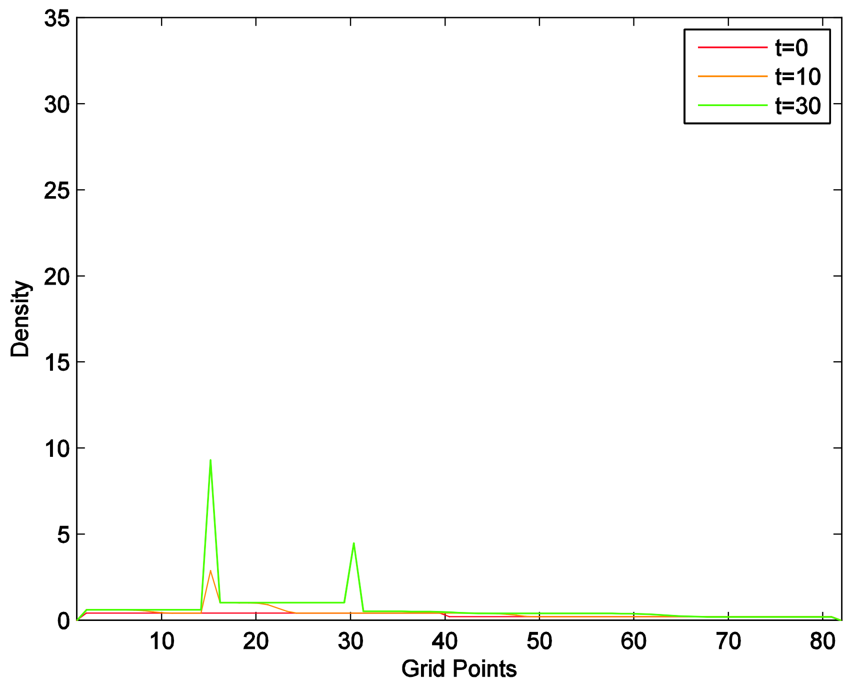
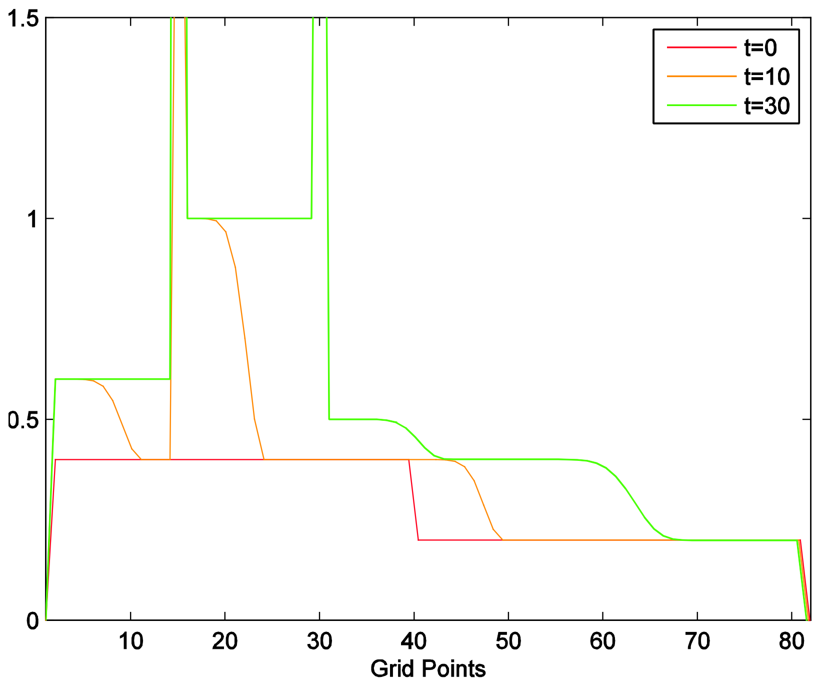
# Interruptions

- Disturbance in the flow
  - Limited bandwidth, link capacity drops to a lower value
  - Router (grid point) is down for some time.

- Example
  - Source node has constant flux of packets below link capacity.
  - One of the grid points has limited bandwidth for a while.
  - One of the inner grid points has a source term.

# Interruptions

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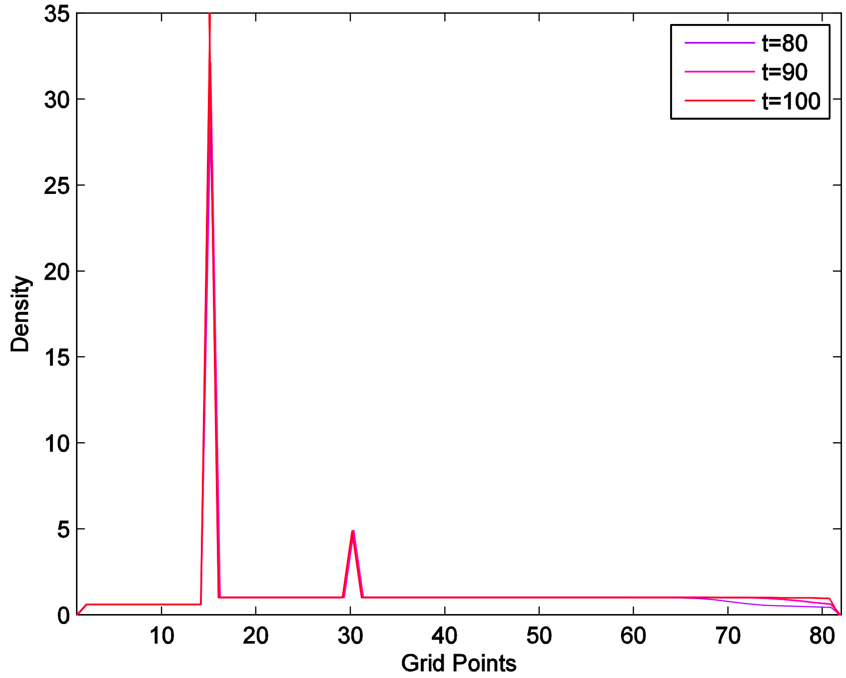
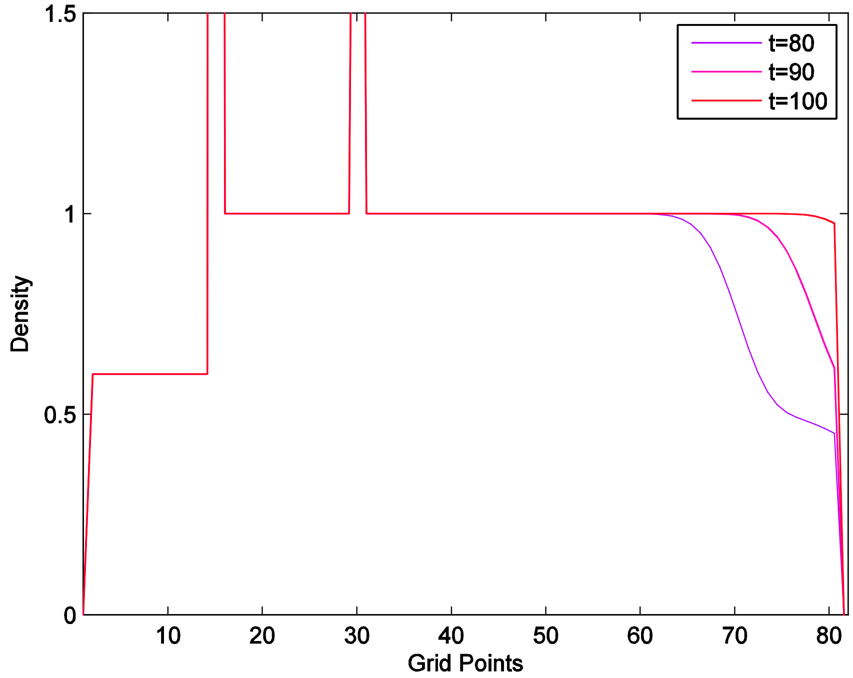
# End of Flow

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# Current Work

- Currently route matrix is static.
- Make route matrix dynamic at every node (changes because of link weight, upstream traffic, etc.

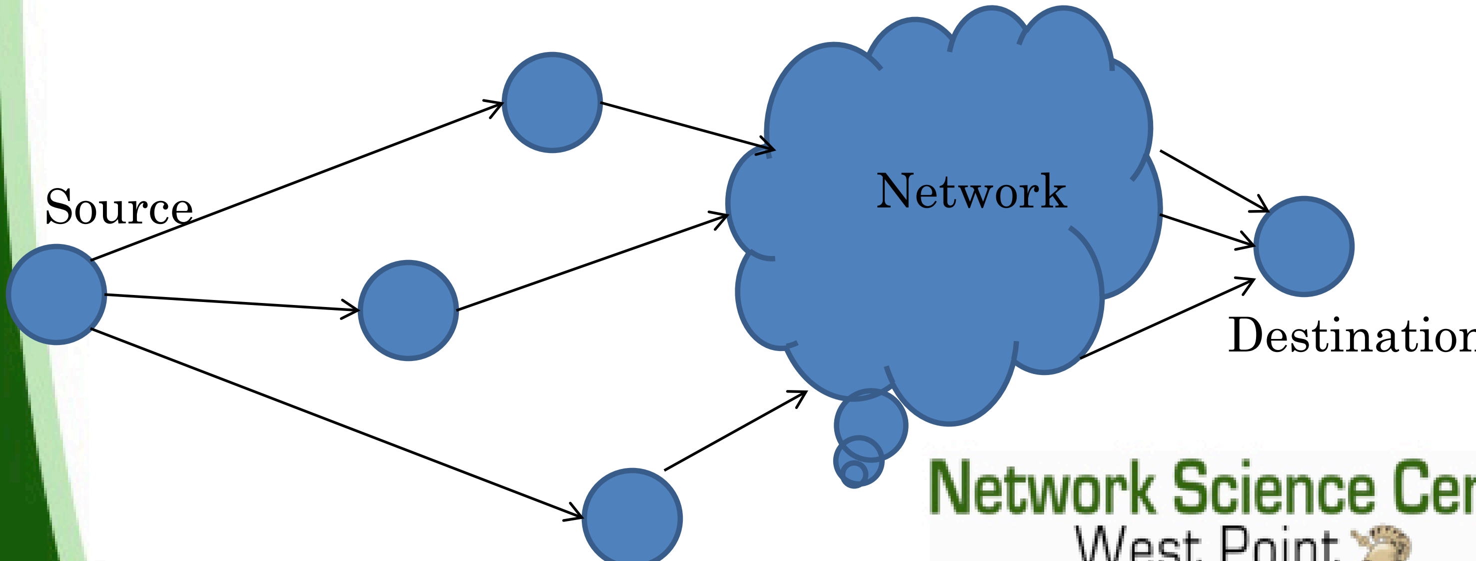
$$R_{j,d} = i$$ **Before**

$$R_{j,d}(t) \neq R_{j,d}(t+1)$$ **Now**

# Current Work

- Also leads to probabilities in which nodes packets will take next.
- Probability of going to node *i* to node *j* with destination *d*.

$$p(j, d, t)$$



# Questions!!!!!!!!!!!!!